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Global existence and asymptotic stability of equilibria to reaction-diffusion systems

Rong-Nian Wang[1] **and Zhong Wei Tang**[2]

[1] Department of Mathematics, NanChang University, NanChang, JiangXi 330031, People's Republic of China
[2] School of Mathematical Sciences, Beijing Normal University, Beijing, 100875, People's Republic of China

E-mail: rnwang@mail.ustc.edu.cn

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Abstract
In this paper, we study weakly coupled reaction-diffusion systems in unbounded domains of $\mathbb{R}^2$ or $\mathbb{R}^3$, where the reaction terms are sums of quasimonotone nondecreasing and nonincreasing functions. Such systems are more complicated than those in many previous publications and little is known about them. A comparison principle and global existence, and boundedness theorems for solutions to these systems are established. Sufficient conditions on the nonlinearities, ensuring the positively Ljapunov stability of the zero solution with respect to H^2-perturbations, are also obtained. As samples of applications, these results are applied to an autocatalytic chemical model and a concrete problem, whose nonlinearities are nonquasimonotone. Our results are novel. In particular, we present a solution to an open problem posed by Escher and Yin (2005 *J. Nonlinear Anal. Theory Methods Appl.* **60** 1065–84).

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1. Introduction

During the past two decades, systems of reaction-diffusion equations have been studied extensively in different contexts and by various methods (see [6, 7, 10, 16, 19, 21, 22] and references therein), motivated both by their widespread occurrence in interacting models of chemical, biological and ecological phenomena, and by the rise of more complicated and challenging issues in the context of coupled PDE systems.

In the study of the chemical basis of morphogenesis, the following reaction-diffusion system, which models certain autocatalytic chemical morphogenetic processes (for example, it explains how patterns might grow from a homogeneous situation and how diffusion of

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morphogenetic chemicals in animals' skins during the growth of an embryo could drive instability), was proposed by Turing [24] (see also [18, 25]):

$$\begin{cases} u_t - d_1\Delta u = av - v^2u & \text{in} \quad (0,\infty)\times\Omega, \\ v_t - d_2\Delta v = v^2u - (a+1)v + f(t,x) & \text{in} \quad (0,\infty)\times\Omega. \end{cases} \tag{1.1}$$

Here, Ω, representing the capacity, is an open and connected domain in $\mathbb{R}^n (n \geqslant 1)$, d_1, d_2, a, c are positive constants, $f(t,x)$ is a positive function defined on $(0,\infty)\times\Omega$, Δ stands for the Laplacian operator with respect to the spatial variable $(x_1,\ldots,x_n) \in \Omega$, $u = u(t,x)$ and $v = v(t,x)$ represent, respectively, the concentrations of two chemical species: an activator and an inhibitor with diffusion rates d_1 and d_2, and hence u and v are nonnegative by their physical interest. The function $f(t,x)$ denotes a chemotactic sensitivity function. Further, the constant a gives the rate at which the concentration of the inhibitor varies from high to low. When Ω is a bounded domain in $\mathbb{R}^n$, system (1.1), under various boundary conditions (of Dirichlet, Neumann or regular oblique derivative type), has been given a thorough study by Hollis *et al* [9], Morgan [15] and Rothe [19] for the global existence and boundedness of solutions. Moreover, Leung and Ortega in [11] proved the existence of periodic solutions to system (1.1) with Dirichlet boundary conditions or regular oblique derivative type boundary conditions. However, in the chemostat, especially in a flow reactor, the selfullage of chemical species cannot be ignored, namely, the following reaction-diffusion system,

$$\begin{cases} u_t - d_1\Delta u = av - cu - v^2u & \text{in} \quad (0,\infty)\times\Omega, \\ v_t - d_2\Delta v = v^2u - (a+1)v + f(t,x) & \text{in} \quad (0,\infty)\times\Omega, \end{cases} \tag{1.2}$$

is often more realistic to describe the interacting process of chemical species, where the constant $c > 0$ corresponds to the selfullage rate of the activator (cf [3, 4]). Biologically or chemically, the most interesting problems in connection with this version of the model are the global existence and the dynamics of nonnegative solutions. However, to our knowledge, so far there have not been any dynamical results on system (1.2). We note that the reaction functions $h(u,v) := av - cu - v^2u$ and $k(u,v) := v^2u - (a+1)v + f(t,x)$ in v and in u, respectively, do not satisfy monotonicity (quasimonotone nonincreasing property, quasimonotone nondecreasing property or mixed quasimonotone property). But in the study of the dynamics of reaction-diffusion equations, the monotonicity of reaction functions often plays an essential role, especially when one tries to establish a comparison principle (see [7, 16, 21]). Of course, this question is more motivated from the mathematical point of view than from the biological one, but it will help us to get more insight into the more extensive class of problems.

Motivated by this problem, we study the following more general and complicated systems of reaction-diffusion equations:

$$\begin{cases} u_t - d_1\Delta u = f_1(u,v) + g_1(u,v) & \text{in } D, \\ v_t - d_2\Delta v = f_2(u,v) + g_2(u,v) & \text{in } D, \\ u(t,x) = v(t,x) = 0 & \text{on } S, \end{cases} \tag{1.3}$$

where d_1, d_2 are positive constants, Ω is an unbounded domain in $\mathbb{R}^n (n = 2 \text{ or } 3)$ with an unbounded inradius $\mathrm{d}(\Omega) := \sup_{x\in\Omega} \mathrm{dist}(x, \partial\Omega)$, e.g. Ω is the whole space $\mathbb{R}^n$, the exterior Ω_e of a bounded domain or the half-space $\mathbb{R}^n_+ = \{x = (x_1,\ldots,x_n) \in \mathbb{R}^n; x_n > 0\}$ and $D := (0,\infty)\times\Omega$, $S := [0,\infty)\times\partial\Omega$. The nonlinearities $f_i, g_i \in C^4(\mathbb{R}^2,\mathbb{R}) (i = 1,2)$ are assumed to satisfy $f_i(0,0) = g_i(0,0) = 0$. Assume further that for fixed u_i, $f_i(u_1,u_2)$ is nondecreasing and $g_i(u_1,u_2)$ is nonincreasing in u_j, $j \neq i$, $i, j = 1, 2$. System (1.3) is a widely used mathematical model for many chemical, physical, biological or ecological phenomena. For details on biological and chemical models involving more general reaction-diffusion systems of type (1.3) we refer to [8, 16, 21].

On the one hand, since the functions f_1 and g_1 possess the monotonicity with respect to only the variable v, the functions f_2 and g_2 possess the monotonicity with respect to only the variable u, and every smooth function with one variable can be written as the sum of a nondecreasing function and a nonincreasing function; the study on systems of type (1.3) is quite useful. On the other hand, if Ω is an unbounded domain, Poincaré's inequality does not hold (cf [23, theorem 2.1]). Hence, one sees from the argument at the end of section 4 in [7] that in this case the principle of linearized stability is not applicable.

In this paper, we first establish a comparison principle corresponding to system (1.3). Then, using the comparison principle together with the abstract stability results developed by Escher and Yin in [6, 7], we obtain the global existence and boundedness theorems for nonnegative solutions to system (1.3). Moreover, we also present sufficient conditions on reaction functions f_i and $g_i (i = 1, 2)$ to ensure the positively Ljapunov stability of the zero solution with respect to H^2-perturbations. As samples of applications, we apply our main results to system (1.2) with $f = 0$ and to a concrete problem, where the nonlinearities are nonquasimonotone. These results are novel.

Remark 1.1. Escher and Yin [7] have used the comparison principle for parabolic systems as presented in [12] and abstract stability results for equilibria of parabolic evolution equations established in [6] to investigate the stability of the zero solution to a special version of system (1.3), which is given in the form

$$\begin{cases} u_t - \Delta u = \Phi(u, v) & \text{in } D, \\ v_t - \Delta v = \Psi(u, v) & \text{in } D, \\ u(t, x) = v(t, x) = 0 & \text{on } S, \end{cases} \tag{1.4}$$

where the nonlinearities $\Phi(u, v)$ and $\Psi(u, v)$ are assumed to satisfy one of the following monotonicity conditions:

(1) $\Phi(u, v)$ is nondecreasing with respect to v and $\Psi(u, v)$ is nondecreasing with respect to u.
(2) $\Phi(u, v)$ is nondecreasing with respect to v and $\Psi(u, v)$ is nonincreasing with respect to u.
(3) $\Phi(u, v)$ is nonincreasing with respect to v and $\Psi(u, v)$ is nonincreasing with respect to u.

Note that our nonlinearities in system (1.3) are more general than those in system (1.4). Furthermore, the comparison principle used in [7] no longer suits system (1.3).

Remark 1.2. The main results in this paper give a solution to the following open problem posed by Escher and Yin in [7, remarks (d)]:

> 'We do not know whether or not the quasimonotonicity of $\Phi(u, v)$ and $\Psi(u, v)$ in [7, theorem 1 and theorem 2] can be relaxed'.

2. Preliminaries

In this section, we introduce some notation, establish some conventions and describe some results which are essential tools in the later sections.

Throughout this paper, $L(X, Y)$ denotes the space of all bounded linear operators from the Banach space X to the Banach space Y with the usual operator norm $\|\cdot\|_{L(X,Y)}$, and $L(X) := L(X, X)$, and $D(A)$ stands for the domain of the linear operator A. We assume that Ω is an unbounded domain in $\mathbb{R}^2$ or in $\mathbb{R}^3$. If the boundary $\partial\Omega$ of Ω is not empty it is assumed to be uniformly C^4-regular; see Browder [2] for its precise definition. We write (1.3) in the abstract form:

$$w_t = A_0 w + G(w), \tag{2.1}$$

where

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad A_0 = \begin{pmatrix} d_1\Delta & 0 \\ 0 & d_2\Delta \end{pmatrix}, \qquad G(w) = \begin{pmatrix} f_1(u,v) + g_1(u,v) \\ f_2(u,v) + g_2(u,v) \end{pmatrix}.$$

For each $i = 1, 2$, set

$$f_i(r,s) = r f_{i1}(r,s) + s f_{i2}(r,s), \qquad g_i(r,s) = r g_{i1}(r,s) + s g_{i2}(r,s), \tag{2.2}$$

for $(r,s) \in \mathbb{R}^2$, where

$$f_{ij}(r,s) = \int_0^1 \partial_j f_i(\sigma r, \sigma s)\,\mathrm{d}\sigma, \qquad g_{ij}(r,s) = \int_0^1 \partial_j g_i(\sigma r, \sigma s)\,\mathrm{d}\sigma, \qquad j = 1, 2.$$

Let us first collect some tools that will frequently be used in the sequel, which may be found in [6] or [7]; for detail, we refer to [1]. Let $H^m(\Omega)$ and $H_0^m(\Omega)$ denote the usual Sobolev spaces based on $L^2(\Omega)$ so that $H^0(\Omega) = L^2(\Omega)$. We identify $L^2(\Omega; \mathbb{R}^2)$ with $L^2(\Omega) \times L^2(\Omega)$. Similarly, we denote by $H^m(\Omega; \mathbb{R}^2)$ the Hilbert space $H^m(\Omega) \times H^m(\Omega)$ with the inner product

$$(w, z)_m = \sum_{|\alpha| \leqslant m} (D^\alpha w, D^\alpha z)_0, \qquad w, z \in H^m(\Omega; \mathbb{R}^2),$$

where $(w,z)_0 = \int_\Omega (w,z)_{\mathbb{R}^2}\,\mathrm{d}x$, and by $H_0^m(\Omega; \mathbb{R}^2)$ the space $H_0^m(\Omega) \times H_0^m(\Omega)$. We also write $C_{\mathrm{BU}}(\overline{\Omega}, \mathbb{R}^2)$ for the Banach space of all bounded and uniformly continuous vector functions $w = (u, v) : \overline{\Omega} \to \mathbb{R}^2$ with the norm

$$\|w\|_{C_{\mathrm{BU}}} := \sup_{x\in\Omega} |w(x)| = \sup_{x\in\Omega} (|u(x)| + |v(x)|).$$

For each $m \in \mathbb{N}$, let $C_{\mathrm{BU}}^m(\overline{\Omega}, \mathbb{R}^2)$ denote the Banach space of all the functions $w \in C_{\mathrm{BU}}(\overline{\Omega}, \mathbb{R}^2)$ which are m times continuously differentiable in $\overline{\Omega}$, with all the derivatives up to the order m in $C_{\mathrm{BU}}(\overline{\Omega}, \mathbb{R}^2)$. They are endowed with the norm

$$\|w\|_{C_{\mathrm{BU}}^m} = \sum_{|\alpha| \leqslant m} \|D^\alpha w(x)\|_{C_{\mathrm{BU}}}.$$

Moreover, given $\beta \in (0, 1)$ and $m \in \mathbb{N}$, let $C_{\mathrm{BU}}^{m+\beta}(\overline{\Omega}, \mathbb{R}^2)$ denote the Banach space of all $w \in C_{\mathrm{BU}}^m(\overline{\Omega}, \mathbb{R}^2)$ such that $D^\alpha w$ with $|\alpha| = m$ are uniformly β-Hölder continuous on $\overline{\Omega}$. The norm in $C_{\mathrm{BU}}^{m+\beta}(\overline{\Omega}, \mathbb{R}^2)$ is given by

$$\|w\|_{C_{\mathrm{BU}}^{m+\beta}} = \|w\|_{C_{\mathrm{BU}}^m} + \sum_{|\alpha|=m} \left(\sup_{x\neq y} \frac{|D^\alpha w(x) - D^\alpha w(y)|}{|x-y|^\beta} \right).$$

We will also use the following Banach spaces:

$$\widehat{C}_{\mathrm{BU}}^{m+\beta}(\overline{\Omega}, \mathbb{R}^2) = \begin{cases} \{w \in C_{\mathrm{BU}}^{m+\beta}(\overline{\Omega}, \mathbb{R}^2);\ w = 0 \text{ on } \partial\Omega\} & \text{if} \quad m = 0, 1, \\ \{w \in C_{\mathrm{BU}}^{m+\beta}(\overline{\Omega}, \mathbb{R}^2);\ w = 0 \text{ and } \Delta w = 0 \text{ on } \partial\Omega\} & \text{if} \quad m = 2, 3. \end{cases}$$

Let us first consider the abstract Cauchy problem (2.1) in $X_0 := L^2(\Omega, \mathbb{R}^2)$, where

$$D(A_0) = H^2(\Omega, \mathbb{R}^2) \cap H_0^1(\Omega, \mathbb{R}^2).$$

Then A_0 is a nonpositive self-adjoint operator in X_0. Hence, $\sigma(A_0) \subset (-\infty, 0]$, A_0 is closed, and A_0 is sectorial. Note that A_0 is the infinitesimal generator of an analytic C_0-semigroup $\{S(t)\}_{t\geqslant 0}$ defined on X_0 (cf [8]). However, since $G(w)$ is, in general, not a mapping from

X_0 into itself, the space X_0 is not suited to (2.1). Here we follow the idea in [7] and hence consider the abstract problem (2.1) in the Hilbert space $X_1 := H^2(\Omega, \mathbb{R}^2) \cap H_0^1(\Omega, \mathbb{R}^2)$. Since $1 \in \rho(A_0)$ and A_0 is closed, for any $w \in X_1$, we may introduce the graph norm in X_1 defined as follows:

$$\|w\|_{X_1} = \|(A_0 - 1)w\|_{X_0}.$$

Clearly, by the open mapping theorem we know that the norm $\|\cdot\|_{X_1}$ is equivalent to the norm $\|\cdot\|_{H^2}$. Now we define the unbounded operator $A_1 : D(A_1) \subset X_1 \to X_1$, which is the restriction of A_0 to X_1 and is given by

$$D(A_1) := D\big(A_0^2\big), \qquad A_1 w := A_0 w \qquad \text{for all} \quad w \in D(A_1).$$

As was shown in [20], we have

$$D(A_1) = \big\{w \in H^4(\Omega, \mathbb{R}^2) \cap H_0^1(\Omega, \mathbb{R}^2);\ A_0 w \in H_0^1(\Omega, \mathbb{R}^2)\big\}.$$

We see that A_1 is a nonpositive self-adjoint operator in X_1 and hence A_1 is a sectorial operator. Similarly, the graph norm of A_1 is equivalent to the norm $\|\cdot\|_{H^4}$. We write $\{T(t)\}_{t\geqslant 0}$ for the analytic semigroup generated by A_1 on X_1. It is not difficult to see that

$$S(t)w = T(t)w, \qquad \text{for} \qquad t \geqslant 0, \quad w \in X_1.$$

Moreover, using arguments similar to those in [7, lemma 1], we can show that $G \in C^1(X_1, X_1)$. Consequently, the standard existence-uniqueness theorems for abstract evolution equations imply that the following abstract Cauchy problem,

$$\begin{cases} w_t = A_1 w + G(w), & t > 0, \\ w(0) = w_0, \end{cases} \tag{2.3}$$

has a unique strong X_1-solution on the maximal interval $[0, T_{\max})$ of existence.

In order to derive suitable *a priori* estimates for solutions to (2.1), we also need to formulate (2.1) in a different functional analytic setting. Define a linear operator A_2 in $X_C := \widehat{C}_{\mathrm{BU}}(\overline{\Omega}, \mathbb{R}^2)$ by

$$A_2 : \quad \widehat{C}_{\mathrm{BU}}^{2+\beta}(\overline{\Omega}, \mathbb{R}^2) \subset X_C \to X_C, \qquad A_2 w := A_0 w \qquad \text{for all} \qquad w \in \widehat{C}_{\mathrm{BU}}^{2+\beta}(\overline{\Omega}, \mathbb{R}^2).$$

Then the operator A_2 is closable and its closure, denoted by A_C, generates an analytic semigroup $\{W(t)\}_{t\geqslant 0}$ on X_C (see [14, theorem 2.4]). At the same time, the domain of A_C can be characterized as

$$D(A_C) = \big\{w \in W_{loc}^{2,p}(\Omega, \mathbb{R}^2),\ p \geqslant 1;\ w, A_0 w \in X_C\big\}.$$

Moreover, it is not difficult to verify that $G \in C^1(X_C, X_C)$. This implies that the abstract Cauchy problem,

$$\begin{cases} w_t = A_C w + G(w), & t > 0, \\ w(0) = w_0, \end{cases} \tag{2.4}$$

has a unique strong X_C-solution on the maximal interval $\big[0, T_{\max}^C\big)$ of existence.

Let $0 < \beta < \frac{1}{2}$. Since the space dimension n is equal to 2 or 3, the Sobolev embedding theorem implies that the imbedding,

$$D(A_0) \hookrightarrow X_C, \quad D(A_1) \hookrightarrow \widehat{C}_{\mathrm{BU}}^{2+\beta}(\overline{\Omega}, \mathbb{R}^2),$$

is continuous (see [1]). Hence we have $D(A_1) \subset D(A_C)$. Consequently, given $w_0 \in D(A_1)$, we can solve (2.3) and (2.4) with the same initial data w_0.

The following proposition establishes the relation between the solutions of (2.3) and (2.4). Proceeding similarly as in the proof of [5, theorem 1], we obtain

Proposition 2.1. *Let an initial datum $w_0 \in D(A_1)$ be given. Then there exists a unique strong solution $w(t) \in C^1([0, T_{\max}), X_1)$ to (2.3) defined on the maximal interval $[0, T_{\max})$ of existence and there exists a unique strong solution $z(t) \in C^1([0, T^C_{\max}), X_C)$ to (2.4) defined on the maximal interval $\left[0, T^C_{\max}\right)$ of existence. Moreover, $T_{\max} = T^C_{\max}$, $w(t) = z(t)$ on $[0, T_{\max})$, and if $T^C_{\max} < \infty$ then*

$$\limsup_{t\to T^C_{\max}} \|z(t)\|_{X_C} = \infty.$$

Definition 2.1 (Escher and Yin [7])**.** *We say that the equilibrium $w = (u, v) = (0, 0)$ of (2.3) is positively Ljapunov stable if it is Ljapunov stable under nonnegative perturbations in $H^2(\Omega, \mathbb{R}^2) \cap H^1_0(\Omega, \mathbb{R}^2)$, i.e. if there is a $\tau > 0$ such that for every $\varepsilon > 0$ there is a $\delta > 0$ with the following property: given $w_0 \in H^2(\Omega, \mathbb{R}^2) \cap H^1_0(\Omega, \mathbb{R}^2)$ with $\|w_0\|_{H^2} \leqslant \delta$ and $w_0 \geqslant 0$, the solution $w(t)$ of (2.3) with the initial datum $w(0) = w_0$ exists globally and satisfies $\|w(t)\|_{H^2} \leqslant \varepsilon$ for all $t \geqslant \tau$.*

The following abstract stability result is proved by Escher and Yin in [6] (see also [7]).

Theorem 2.1. *Let $w_0 \in X_1$ and let $T_{\max}(w_0)$ be the maximal existence time of the corresponding solution w to (2.3) with the initial datum w_0. Assume that:*

(1) There exists a δ_0 such that $T_{\max}(w_0) = \infty$, if $\|w_0\|_{X_1} \leqslant \delta_0$.
(2) There exists a $M > 0$ such that

$$\|w(t + 2)\|_{X_1} \leqslant M\|w(t)\|_{X_0}, \qquad \forall\, t \geqslant 0.$$

(3) There exists $\varepsilon_1 > 0$ such that

$$(w(t), G(w(t) + A_0 w(t)))_0 \leqslant 0, \quad \text{if } \|w(t)\|_{X_1} \leqslant \varepsilon_1.$$

Then $(0, 0)$ is X_1-Ljapunov stable.

3. The main results and their proofs

In order to prove the desired results, we first give the following important comparison results for the nonquasimonotone coupled system (1.3). Let $D_T = (0, T] \times \Omega$, $S_T = (0, T] \times \partial\Omega$, where $T > 0$ is any constant.

Theorem 3.1 (Comparison principle)**.** *Assume that there is a 6-tuple $\mathbf{w} = \{\widehat{u}, \widehat{v}, u, v, \widetilde{u}, \widetilde{v}\}$ of functions on D_T such that $\mathbf{w}$ is bounded and continuous on $\overline{D}_T$ and $(\widehat{u}, \widehat{v}) \leqslant (\widetilde{u}, \widetilde{v})$ in D_T. Let*

$$\rho_M = \max\Big\{ \sup_{(t,x)\in D_T} \widetilde{u}(t, x),\ \sup_{(t,x)\in D_T} u(t, x)\Big\},$$
$$\sigma_M = \max\Big\{ \sup_{(t,x)\in D_T} \widetilde{v}(t, x),\ \sup_{(t,x)\in D_T} v(t, x)\Big\},$$
$$\rho_m = \min\Big\{ \inf_{(t,x)\in D_T} \widehat{u}(t, x),\ \inf_{(t,x)\in D_T} u(t, x)\Big\},$$
$$\sigma_m = \min\Big\{ \inf_{(t,x)\in D_T} \widehat{v}(t, x),\ \inf_{(t,x)\in D_T} v(t, x)\Big\},$$

and let I_1 and I_2 be the intervals such that $I_1 \supset (\rho_m, \rho_M), I_2 \supset (\sigma_m, \sigma_M)$. Moreover, assume that $f_1(u, v)\,(g_1(u, v))$ is nondecreasing (nonincreasing) in I_2 for all $u \in I_1$, $f_2(u, v)\,(g_2(u, v))$ is nondecreasing (nonincreasing) in I_1 for all $v \in I_2$ and

$$\begin{cases} \widetilde{u}_t - d_1\Delta\widetilde{u} - f_1(\widetilde{u}, \widetilde{v}) - g_1(\widetilde{u}, \widehat{v}) \geqslant u_t - d_1\Delta u - f_1(u, v) - g_1(u, v) \\ \quad \geqslant \widehat{u}_t - d_1\Delta\widehat{u} - f_1(\widehat{u}, \widehat{v}) - g_1(\widehat{u}, \widetilde{v}) \quad (t, x) \in D_T, \\ \widetilde{v}_t - d_2\Delta\widetilde{v} - f_2(\widetilde{u}, \widetilde{v}) - g_2(\widehat{u}, \widetilde{v}) \geqslant v_t - d_2\Delta v - f_2(u, v) - g_2(u, v) \\ \quad \geqslant \widehat{v}_t - d_2\Delta\widehat{v} - f_2(\widehat{u}, \widehat{v}) - g_2(\widetilde{u}, \widehat{v}) \quad (t, x) \in D_T, \\ (\widehat{u}(t, x), \widehat{v}(t, x)) \leqslant (u(t, x), v(t, x)) \leqslant (\widetilde{u}(t, x), \widetilde{v}(t, x)) \quad (t, x) \in S_T, \\ (\widehat{u}(0, x), \widehat{v}(0, x)) \leqslant (u(0, x), v(0, x)) \leqslant (\widetilde{u}(0, x), \widetilde{v}(0, x)) \quad x \in \Omega. \end{cases} \tag{3.1}$$

Then 6-tuple $\mathbf{w}$ *of functions satisfies the following relation:*

$$(\widehat{u}(t, x), \widehat{v}(t, x)) \leqslant (u(t, x), v(t, x)) \leqslant (\widetilde{u}(t, x), \widetilde{v}(t, x)) \quad (t, x) \in \overline{D}_T.$$

Proof. Let $(M_1, M_2) \geqslant (0, 0)$ be any constant vector such that

$$(M_1, M_2) \geqslant (\widetilde{u}(t, x), \widetilde{v}(t, x)) \quad \text{on } \overline{D}_T,$$

and let $(u_3, u_4) = (M_1 - u_1, M_2 - u_2)$. Consider the following extended system of 4-equalities:

$$\begin{cases} u_{1t} - d_1\Delta u_1 = f_1(u_1, u_2) + g_1(u_1, M_2 - u_4), \\ u_{3t} - d_1\Delta u_3 = -f_1(M_1 - u_3, M_2 - u_4) - g_1(M_1 - u_3, u_2), \\ u_{2t} - d_2\Delta u_2 = f_2(u_1, u_2) + g_2(M_1 - u_3, u_2), \\ u_{4t} - d_2\Delta u_4 = -f_2(M_1 - u_3, M_2 - u_4) - g_2(u_1, M_2 - u_4). \end{cases}$$

Define

$$\begin{aligned} F_1(u_1, u_2, u_3, u_4) &:= f_1(u_1, u_2) + g_1(u_1, M_2 - u_4), \\ F_3(u_1, u_2, u_3, u_4) &:= -f_1(M_1 - u_3, M_2 - u_4) - g_1(M_1 - u_3, u_2), \\ F_2(u_1, u_2, u_3, u_4) &:= f_2(u_1, u_2) + g_2(M_1 - u_3, u_2), \\ F_4(u_1, u_2, u_3, u_4) &:= -f_2(M_1 - u_3, M_2 - u_4) - g_2(u_1, M_2 - u_4). \end{aligned}$$

It is easily seen from the quasimonotone nondecreasing property of $f_i (i = 1, 2)$ and the quasimonotone nonincreasing property of $g_i (i = 1, 2)$ that for each $i = 1, \ldots, 4$, F_i is quasimonotone nondecreasing, i.e., for fixed u_i, $F_i(u_1, u_2, u_3, u_4)$ is nondecreasing in $u_j, j \neq i, i, j = 1, \ldots, 4$. Moreover, a direct computation shows that the vector $(p_1, p_2, p_3, p_4) := (\widetilde{u}, \widetilde{v}, M_1 - \widehat{u}, M_2 - \widehat{v})$ satisfies

$$\begin{aligned} & p_{1t} - d_1\Delta p_1 - F_1(p_1, p_2, p_3, p_4) \geqslant u_t - d_1\Delta u - F_1(u, v, M_1 - u, M_2 - v), \quad (t, x) \in D_T, \\ & p_{2t} - d_2\Delta p_2 - F_2(p_1, p_2, p_3, p_4) \geqslant v_t - d_2\Delta u - F_2(u, v, M_1 - u, M_2 - v), \quad (t, x) \in D_T, \\ & p_{3t} - d_1\Delta p_3 - F_3(p_1, p_2, p_3, p_4) \geqslant -u_t + d_1\Delta(M_1 - u) - F_3(u, v, M_1 - u, M_2 - v), \\ & \qquad (t, x) \in D_T, \\ & p_{4t} - d_2\Delta p_4 - F_4(p_1, p_2, p_3, p_4) \geqslant -v_t + d_2\Delta(M_2 - v) - F_4(u, v, M_1 - u, M_2 - v), \\ & \qquad (t, x) \in D_T, \\ & (p_1, p_2, p_3, p_4) \geqslant (u, v, M_1 - u, M_2 - v), \quad (t, x) \in S_T, \\ & (p_1(0, x), p_2(0, x), p_3(0, x), p_4(0, x)) \geqslant (u(0, x), v(0, x), M_1 - u(0, x), M_2 - v(0, x)), \\ & \qquad x \in \Omega. \end{aligned}$$

Now let $q_i = p_i - r_i$, where $(r_1, r_2, r_3, r_4) := (u, v, M_1 - u, M_2 - v)$. Then we have

$$\begin{cases} q_{it} - d_i\Delta q_i \geqslant F_i(p_1, p_2, p_3, p_4) - F_i(r_1, r_2, r_3, r_4) \\ \quad = \displaystyle\sum_{j=1}^{4} \frac{\partial F_i(\tau_1, \tau_2, \tau_3, \tau_4)}{\partial u_j} q_j, \quad (t, x) \in D_T, \\ q_i(t, x) \geqslant 0, \quad (t, x) \in S_T, \\ q_i(0, x) \geqslant 0, \quad x \in \Omega, \quad (i = 1, \ldots, 4), \end{cases}$$

where $d_3 = d_1, d_4 = d_2$ and $(\tau_1, \tau_2, \tau_3, \tau_4)$ is an intermediate value between (p_1, p_2, p_3, p_4) and (r_1, r_2, r_3, r_4). The quasimonotone nondecreasing property of F_i implies that

$$c_{ij} \equiv \frac{\partial F_i(\tau_1, \tau_2, \tau_3, \tau_4)}{\partial u_j} \geqslant 0 \quad (i \neq j, i, j = 1, \dots, 4).$$

Moreover, the smoothness assumption on F_i and boundedness assumption on $\mathbf{w}$ ensure that c_{ij} are bounded on $\overline{D}_T$ for all $i, j = 1, \dots, 4$. Then it follows from a direct application of [17, lemma 5.2] that

$$\mathbf{q} := (q_1, \dots, q_4) \geqslant 0 \qquad \text{on} \quad \overline{D}_T,$$

i.e.

$$(p_1, p_2, p_3, p_4) \geqslant (r_1, r_2, r_3, r_4) \qquad \text{on} \quad \overline{D}_T.$$

This yields

$$(\widetilde{u}, \widetilde{v}, M_1 - \widehat{u}, M_2 - \widehat{v}) \geqslant (u, v, M_1 - u, M_2 - v) \qquad \text{on} \quad \overline{D}_T.$$

Hence, $(\widehat{u}(t,x), \widehat{v}(t,x)) \leqslant (u(t,x), v(t,x)) \leqslant (\widetilde{u}(t,x), \widetilde{v}(t,x))$ on $\overline{D}_T$. The proof is complete. □

Remark 3.1. Theorem 3.1 will be an important tool for showing the boundedness of the solutions of (2.3) on the maximal interval $[0, T_{\max})$ of existence (see the proof of theorem 3.2).

In the following, given $w_0 \in D(A_1)$, we denote the maximal existence time of (2.3) by $T_{\max}(w_0)$. The local existence and uniqueness of solution to (2.3) follow from the functional analytic frames and proposition 2.1 in section 2. Now let $w(t) = (u(t), v(t)) \in C^1([0, T_{\max}), X_1)$ be a strong solution of (2.3) with $w(0) = w_0 \in X_1$. Let T be any given constant such that $T < T_{\max}$, then by proposition 2.1 we have $w(t) \in C^1([0, T], X_C)$. Hence, it follows from the property of space X_C that w is bounded on $\overline{D}_T$. Let I_1' and I_2' be the intervals $\left(\inf_{(t,x)\in D_T} u(t,x), \sup_{(t,x)\in D_T} u(t,x)\right)$ and $\left(\inf_{(t,x)\in D_T} v(t,x), \sup_{(t,x)\in D_T} v(t,x)\right)$, respectively.

The global existence and boundedness of solutions to (2.3) are given in the following theorem.

Theorem 3.2. *Assume that*

(1) (1.3) has a coupled upper bound (η_1, η_2) in relation to $(0, 0)$, i.e., (η_1, η_2) is a constant vector with $\eta_i > 0\,(i = 1, 2)$ satisfying

$$f_1(\eta_1, \eta_2) + \eta_1 g_{11}(\eta_1, 0) \leqslant 0, \qquad f_2(\eta_1, \eta_2) + \eta_2 g_{22}(0, \eta_2) \leqslant 0,$$
$$g_{12}(0, \eta_2) \geqslant 0, \qquad g_{21}(\eta_1, 0) \geqslant 0,$$

and

(2) $f_1(u, v)\,(g_1(u, v))$ is nondecreasing (nonincreasing) in $I_2' \cup (0, \eta_2)$ for all $u \in I_1' \cup (0, \eta_1)$ and $f_2(u, v)\,(g_2(u, v))$ is nondecreasing (nonincreasing) in $I_1' \cup (0, \eta_1)$ for all $v \in I_2' \cup (0, \eta_2)$.

Then there exist constants $\delta_0 > 0$ and $M > 0$ such that (2.3) has a unique strong solution $w(t)$ with $w(t) \geqslant (0, 0)$, which is defined for all time $t \geqslant 0$, namely, $T_{\max}(w_0) = \infty$, and

$$\sup_{t\geqslant 0} \|w(t)\|_{X_C} \leqslant M, \tag{3.2}$$

provided $w_0 \geqslant (0, 0)$ and $\|w_0\|_{X_1} \leqslant \delta_0$.

Proof. Set

$$\delta_0 := {M_0}^{-1} \min\{\eta_1, \eta_2\},$$

where M_0 is the embedding constant from the Sobolev space X_1 to X_C, i.e. M_0 is a positive constant such that

$$\|w\|_{X_C} \leqslant M_0\|w\|_{X_1}, \tag{3.3}$$

for all $w \in X_1$.

Choosing $w_0 \in D(A_1)$ with $w_0 \geqslant 0$ and $\|w\|_{X_1} \leqslant \delta_0$, then we have

$$(\eta_1, \eta_2) \geqslant w(0) \geqslant (0, 0).$$

Since (η_1, η_2) is a coupled upper bound in relation to $(0, 0)$, it follows that (η_1, η_2) and $(0, 0)$ satisfy the inequalities in (3.1), with $(\widehat{u}, \widehat{v})$ and $(\widetilde{u}, \widetilde{v})$ replaced by $(0, 0)$ and (η_1, η_2), respectively. An application of theorem 3.1 shows that

$$(\eta_1, \eta_2) \geqslant w(t, x) \geqslant (0, 0), \tag{3.4}$$

for all $(t, x) \in \overline{D}_T (= [0, T] \times \overline{\Omega})$. Hence, it follows from the arbitrariness of $T (<T_{\max})$ that the inequality (3.4) holds for all $(t, x) \in [0, T_{\max}(w_0)) \times \overline{\Omega}$. Moreover, in view of proposition 2.1, it follows from a standard continuation argument that there exists a constant $M \geqslant 0$ such that

$$T_{\max}(w_0) = \infty, \qquad \text{and} \qquad \sup_{t\geqslant 0} \|w(t)\|_{X_C} \leqslant M,$$

provided $w_0 \geqslant 0$ and $\|w\|_{X_1} \leqslant \delta_0$. This completes the proof. □

Now we are in a position to present our stability results:

Theorem 3.3. *Assume that*

(1) the hypotheses (1) *and* (2) *in theorem 3.2 hold, and*
(2) there exists $\varepsilon_1 > 0$ *such that*

$$(w(t), G(w(t)) + A_0 w(t))_0 \leqslant 0, \qquad if \quad \|w(t)\|_{X_1} \leqslant \varepsilon_1. \tag{3.5}$$

Then the trivial solution $w = (0, 0)$ *of (1.3) is positively Ljapunov stable.*

Proof. Let $w(t) = (u(t), v(t)) \in C^1([0, \infty), X_1)$ be the strong solution with $w(0) = w_0$.

From the decomposition (2.2) we can write

$$G(w(t)) = A(t)w(t), \quad t \geqslant 0, \tag{3.6}$$

where operators $A(t)$ are given by

$$A(t) = \begin{pmatrix} f_{11}(u(t), v(t)) + g_{11}(u(t), v(t)) & f_{12}(u(t), v(t)) + g_{12}(u(t), v(t)) \\ f_{21}(u(t), v(t)) + g_{21}(u(t), v(t)) & f_{22}(u(t), v(t)) + g_{22}(u(t), v(t)) \end{pmatrix}, \quad t \geqslant 0.$$

Since $w(t) = (u(t), v(t))$ is uniformly bounded by (3.2) and operators $A(t)$ carry a symmetric structure we have $A \in C^1(\mathbb{R}^+, L(X_0))$ and there exists a constant M_1 such that

$$\|A(t)\|_{L(X_0)} \leqslant M_1 \quad \text{for} \quad t \geqslant 0. \tag{3.7}$$

At the same time, since A_C is the generator of the analytic semigroup $\{W(t)\}_{t\geqslant 0}$ on X_C and the Fré chet derivative of $G \in C^1(X_C, X_C)$ is bounded on bounded subsets of X_C, we have the following *a priori* estimate:

$$\left\| \frac{\mathrm{d}w(t)}{\mathrm{d}t} \right\|_{X_C} \leqslant M_2 \quad \text{for} \quad t \geqslant 0, \tag{3.8}$$

where M_2 is a positive constant (cf [5, Proposition 4.1]). Note further that

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix},$$

where

$$a_{11}(t) = \partial_1 f_{11}(u(t), v(t))u'(t) + \partial_1 g_{11}(u(t), v(t))u'(t) \\ + \partial_2 f_{11}(u(t), v(t))v'(t) + \partial_2 g_{11}(u(t), v(t))v'(t),$$

$$a_{12}(t) = \partial_1 f_{12}(u(t), v(t))u'(t) + \partial_1 g_{12}(u(t), v(t))u'(t) \\ + \partial_2 f_{12}(u(t), v(t))v'(t) + \partial_2 g_{12}(u(t), v(t))v'(t),$$

$$a_{21}(t) = \partial_1 f_{21}(u(t), v(t))u'(t) + \partial_1 g_{21}(u(t), v(t))u'(t) \\ + \partial_2 f_{21}(u(t), v(t))v'(t) + \partial_2 g_{21}(u(t), v(t))v'(t),$$

$$a_{22}(t) = \partial_1 f_{22}(u(t), v(t))u'(t) + \partial_1 g_{22}(u(t), v(t))u'(t) \\ + \partial_2 f_{22}(u(t), v(t))v'(t) + \partial_2 g_{22}(u(t), v(t))v'(t),$$

for $t \geqslant 0$. Consequently, using (3.2) and (3.8), we deduce that there is a constant $M_3 > 0$ such that

$$\left\| \frac{\mathrm{d}A(t)}{\mathrm{d}t} \right\|_{L(X_0)} \leqslant M_3 \quad \text{for} \quad t \geqslant 0. \tag{3.9}$$

Moreover, it follows from (3.7) and (3.9) that

$$\|A(t)\|_{L(X_0)} + \left\| \frac{\mathrm{d}A(t)}{\mathrm{d}t} \right\|_{L(X_0)} \leqslant M_4 \quad \text{for} \quad t \geqslant 0, \tag{3.10}$$

where M_4 is a positive constant.

Consider the following homogeneous Cauchy problem,

$$q_t = (A_0 + A(t))q, \qquad t > 0. \tag{3.11}$$

Because $w(t)$ is the unique strong solution to (2.3), it follows that $w(t)$ is also a mild solution to (3.11). Moreover,

$$w(t) = U(t, s)w(s) \quad \text{for} \quad t \in [s, s+2], \tag{3.12}$$

where $\{U(t, s); 0 \leqslant s \leqslant t \leqslant s + 2\}$ is the evolution system associated with the operator $\{A_0 + A(t); t \in [s, s+2]\}$. It follows from (3.10) and [6, proof of lemma 3.4] that there exists a constant M_5, which is independent of s, such that

$$\max_{t\in[s,s+2]} \|U(t, s)\|_{L(X_0)} \leqslant M_5.$$

Hence combining this with (3.12), we find

$$\|w(t)\|_{X_0} \leqslant M_5\|w(s)\|_{X_0}, \qquad t \in [s, s+2]. \tag{3.13}$$

Moreover, using (3.10), by the arguments similar to those in [5], we can show that there exists a constant $M_6 > 0$ such that

$$\|A_0 w(s+2)\|_{X_0} \leqslant M_6 \max_{t\in[s,s+2]} \|w(s)\|_{X_0}.$$

Using (3.13) we have

$$\|A_0 w(s+2)\|_{X_0} \leqslant M_7\|w(s)\|_{X_0}, \qquad s \geqslant 0.$$

Since the norm $\|\cdot\|_{X_1}$ is equivalent to the norm $\|\cdot\|_{H^2}$, it follows that there exists a $K_0 > 0$ such that

$$\|w(s+2)\|_{X_1} \leqslant K_0\|w(s)\|_{X_0}, \qquad s \geqslant 0. \tag{3.14}$$

Finally, in view of theorem 2.1, by theorem 3.2 and the estimates (3.5) and (3.14), we obtain the assertion of the theorem. □

Remark 3.2. The main idea of the proof of theorem 3.3 comes from the nice proof of [7, lemma 6].

The following theorem gives some sufficient conditions on the nonlinearities, which ensure the positively Ljapunov stability of the zero solution to (1.3).

Theorem 3.4. *Let the hypotheses* (1) *and* (2) *in theorem 3.2 hold. In addition, let us assume that there exists a constant* $\varepsilon_0 > 0$ *such that one of the following conditions is satisfied:*

(H_1) $f_{11}(w) < 0, g_{11}(w) < 0, \quad 4f_{11}(w)f_{22}(w) - (f_{12}(w) + f_{21}(w))^2 \geqslant 0$ *and* $g_{11}(w)g_{22}(w) - (g_{12}(w) + g_{21}(w))^2 \geqslant 0,$

(H_2) $f_{22}(w) < 0, \quad g_{11}(w) < 0, \quad 4f_{11}(w)f_{22}(w) - (f_{12}(w) + f_{21}(w))^2 \geqslant 0$ *and* $g_{11}(w)g_{22}(w) - (g_{12}(w) + g_{21}(w))^2 \geqslant 0,$

(H_3) $f_{11}(w) < 0, \quad g_{22}(w) < 0, \quad 4f_{11}(w)f_{22}(w) - (f_{12}(w) + f_{21}(w))^2 \geqslant 0$ *and* $g_{11}(w)g_{22}(w) - (g_{12}(w) + g_{21}(w))^2 \geqslant 0,$

(H_4) $f_{22}(w) < 0, \quad g_{22}(w) < 0, \quad 4f_{11}(w)f_{22}(w) - (f_{12}(w) + f_{21}(w))^2 \geqslant 0$ *and* $g_{11}(w)g_{22}(w) - (g_{12}(w) + g_{21}(w))^2 \geqslant 0,$

(H_5) $f_{11}(w) < 0, \quad g_{11}(w) < 0, \quad 4f_{11}(w)g_{22}(w) - (f_{21}(w) + g_{12}(w))^2 \geqslant 0$ *and* $g_{11}(w)f_{22}(w) - (g_{21}(w) + f_{12}(w))^2 \geqslant 0,$

(H_6) $f_{22}(w) < 0, \quad g_{11}(w) < 0, \quad 4f_{11}(w)g_{22}(w) - (f_{21}(w) + g_{12}(w))^2 \geqslant 0$ *and* $g_{11}(w)f_{22}(w) - (g_{21}(w) + f_{12}(w))^2 \geqslant 0,$

(H_7) $f_{11}(w) < 0, \quad g_{22}(w) < 0, \quad 4f_{11}(w)g_{22}(w) - (f_{21}(w) + g_{12}(w))^2 \geqslant 0$ *and* $g_{11}(w)f_{22}(w) - (g_{21}(w) + f_{12}(w))^2 \geqslant 0,$

(H_8) $f_{22}(w) < 0, \quad g_{22}(w) < 0, \quad 4f_{11}(w)g_{22}(w) - (f_{21}(w) + g_{12}(w))^2 \geqslant 0$ *and* $g_{11}(w)f_{22}(w) - (g_{21}(w) + f_{12}(w))^2 \geqslant 0,$

(H_9) $f_{11}(w), \quad f_{22}(w), \quad g_{11}(w), \quad g_{22}(w) \leqslant 0, \quad f_{12}(w) + f_{21}(w) = 0$ *and* $g_{12}(w) + g_{21}(w) = 0,$

(H_{10}) $f_{22}(w) < 0, \quad 4f_{11}(w)f_{22}(w) - (f_{12}(w) + f_{21}(w))^2 \geqslant 0, \quad g_{11}(w) \leqslant 0,$ $g_{22}(w) \leqslant 0$ *and* $g_{12}(w) + g_{21}(w) = 0,$

(H_{11}) $f_{11}(w) < 0, \quad 4f_{11}(w)f_{22}(w) - (f_{12}(w) + f_{21}(w))^2 \geqslant 0, \quad g_{11}(w) \leqslant 0,$ $g_{22}(w) \leqslant 0$ *and* $g_{12}(w) + g_{21}(w) = 0,$

(H_{12}) $g_{22}(w) < 0, \quad 4g_{11}(w)g_{22}(w) - (g_{12}(w) + g_{21}(w))^2 \geqslant 0, \quad f_{11}(w) \leqslant 0,$ $f_{22}(w) \leqslant 0$ *and* $f_{12}(w) + f_{21}(w) = 0,$

(H_{13}) $g_{11}(w) < 0, \quad 4g_{11}(w)g_{22}(w) - (g_{12}(w) + g_{21}(w))^2 \geqslant 0, \quad f_{11}(w) \leqslant 0,$ $f_{22}(w) \leqslant 0$ *and* $f_{12}(w) + f_{21}(w) = 0,$

provided $w \geqslant 0$ *and* $|w| \leqslant \varepsilon_0$. *Then the trivial solution* $w = (0, 0)$ *of (1.3) is positively Ljapunov stable.*

Proof. We prove that if one of hypotheses (H_1), (H_5), (H_9) and (H_{11}) holds, then we have

$$(y, G(y) + A_0 y)_0 \leqslant 0, \tag{3.15}$$

provided $y \geqslant 0$ and $|y| \leqslant \varepsilon_0$. The assertion (3.15), under other hypotheses of theorem, may be obtained by a similar argument.

Let $y = (y_1, y_2) \in X_1$. By (2.3) and the decomposition (2.2), a straightforward calculation yields:

$$\begin{aligned}(y, G(y) + A_0 y)_0 &= (y_1, \Delta y_1)_0 + (y_2, \Delta y_2)_0 + (y_1, f_1(y_1, y_2))_0 + (y_1, g_1(y_1, y_2))_0 \\ &\quad + (y_2, f_2(y_1, y_2))_0 + (y_2, g_2(y_1, y_2))_0 \\ &= -\|y_1\|^2_{H_0^1} - \|y_2\|^2_{H_0^1} + (y_1, y_1 f_{11}(y_1, y_2) + y_2 f_{12}(y_1, y_2))_0 \\ &\quad + (y_2, y_1 f_{21}(y_1, y_2) + y_2 f_{22}(y_1, y_2))_0 + (y_1, y_1 g_{11}(y_1, y_2))_0 \\ &\quad + (y_1, y_2 g_{12}(y_1, y_2))_0 + (y_2, y_1 g_{21}(y_1, y_2) + y_2 g_{22}(y_1, y_2))_0.\end{aligned}$$

Let the hypothesis (H_1) hold. We write $(y, G(y) + A_0 y)_0$ in the form

$$(y, G(y) + A_0 y)_0 = -\|y_1\|^2_{H_0^1} - \|y_2\|^2_{H_0^1} + a_0 + a_1 + b_0 + b_1, \tag{3.16}$$

where

$$a_0 = \left(f_{11}, \left(y_1 + \frac{f_{12} + f_{21}}{2 f_{11}} y_2\right)^2\right)_0, \qquad a_1 = \left(\frac{4 f_{11} f_{22} - (f_{12} + f_{21})^2}{4 f_{11}}, y_2^2\right)_0,$$

$$b_0 = \left(g_{11}, \left(y_1 + \frac{g_{12} + g_{21}}{2 g_{11}} y_2\right)^2\right)_0, \qquad b_1 = \left(\frac{4 g_{11} g_{22} - (g_{12} + g_{21})^2}{4 g_{11}}, y_2^2\right)_0.$$

Then it follows that

$$a_i \leqslant 0 \qquad \text{and} \qquad b_i \leqslant 0, \quad i = 0, 1,$$

and hence

$$(y, G(y) + A_0 y)_0 \leqslant a_0 + a_1 + b_0 + b_1 \leqslant 0,$$

provided $y \geqslant 0$ and $|y| \leqslant \varepsilon_0$. If the hypothesis (H_5) holds, then we write $(y, G(y) + A_0 y)_0$ in the form

$$(y, G(y) + A_0 y)_0 = -\|y_1\|^2_{H_0^1} - \|y_2\|^2_{H_0^1} + c_0 + c_1 + d_0 + d_1,$$

where

$$c_0 = \left(f_{11}, \left(y_1 + \frac{f_{21} + g_{12}}{2 f_{11}} y_2\right)^2\right)_0, \qquad c_1 = \left(\frac{4 f_{11} g_{22} - (f_{21} + g_{12})^2}{4 f_{11}}, y_2^2\right)_0,$$

$$d_0 = \left(g_{11}, \left(y_1 + \frac{g_{21} + f_{12}}{2 g_{11}} y_2\right)^2\right)_0, \qquad d_1 = \left(\frac{4 g_{11} f_{22} - (g_{21} + f_{12})^2}{4 g_{11}}, y_2^2\right)_0.$$

It follows that

$$c_i \leqslant 0 \qquad \text{and} \qquad d_i \leqslant 0, \quad i = 0, 1,$$

and hence

$$(y, G(y) + A_0 y)_0 \leqslant c_0 + c_1 + d_0 + d_1 \leqslant 0,$$

provided $y \geqslant 0$ and $|y| \leqslant \varepsilon_0$.

Let the hypothesis (H_9) hold, then

$$\begin{aligned}(y, G(y) + A_0 y)_0 &\leqslant -\|y_1\|^2_{H_0^1} - \|y_2\|^2_{H_0^1} + (y_1, y_2 f_{12}(y_1, y_2))_0 + (y_2, y_1 f_{21}(y_1, y_2))_0 \\ &\quad + (y_1, y_2 g_{12}(y_1, y_2))_0 + (y_2, y_1 g_{21}(y_1, y_2))_0 \leqslant 0,\end{aligned}$$

provided $y \geqslant 0$ and $|y| \leqslant \varepsilon_0$.

Finally, if the hypothesis (H_{11}) holds, then we have

$$(y, G(y) + A_0 y)_0 \leqslant a_0 + a_1 + (y_1, y_2 g_{12}(y_1, y_2))_0 + (y_2, y_1 g_{21}(y_1, y_2))_0 \leqslant 0,$$

provided $y \geqslant 0$ and $|y| \leqslant \varepsilon_0$, where $a_i (i = 1, 2)$ are functions appearing in (3.16).

Hence by (3.3), one of the hypotheses $(H_1) - (H_{13})$ ensures

$$(y, G(y) + A_0 y)_0 \leqslant 0, \tag{3.17}$$

provided $y \geqslant 0$ and $\|y\|_{X_1} \leqslant \varepsilon_1$, where $\varepsilon_1 := \varepsilon_0/M_0$.

Finally, combining theorem 2.1 and the estimates (3.2), (3.14) and (3.17), we complete the proof. □

Remark 3.3.

(1) One sees that theorem 3.1 (the comparison principle) established by us plays a crucial role in showing the global existence and the asymptotic behavior of the solution.
(2) Theorems 3.3 and 3.4 cover recent results in [6, 7].
(3) The results of theorems 3.2, 3.3 and 3.4 are also true for bounded domains in $\mathbb{R}^n (n \geqslant 1)$ with a smooth boundary.

4. Applications

In this section, we apply our main results to a Brusselator problem and to a concrete example. The global existence of solutions is proved. Furthermore, we show that in both cases the trivial solution (0, 0) is positively Ljapunov stable.

Example 1. Consider the reduced Brusselator problem,

$$\begin{cases} u_t - d_1 \Delta u = av - cu - v^2 u & \text{in } D, \\ v_t - d_2 \Delta v = v^2 u - (a+1)v & \text{in } D, \\ u(t, x) = v(t, x) = 0 & \text{on } S, \\ u(0, x) = u_0(x) \geqslant 0, \quad v(0, x) = v_0(x) \geqslant 0 & \text{in } \Omega, \end{cases} \tag{4.1}$$

where d_1, d_2, a, c are positive constants and satisfy the following condition:

$$\frac{a^2}{a+1} < 4c. \tag{4.2}$$

Let us denote

$$f_1(u, v) = av - cu, \qquad g_1(u, v) = -v^2 u,$$
$$f_2(u, v) = v^2 u, \qquad g_2(u, v) = -(a+1)v.$$

Then for each $i = 1, 2$, $f_i(u, v)$ is quasimonotone nondecreasing in $\mathbb{R}^+$, $g_i(u, v)$ is quasimonotone nonincreasing in $\mathbb{R}^+$, namely, the functions $f_i, g_i, i = 1, 2$, satisfy the hypothesis (1) in theorem 3.2, and the decomposition (2.2) implies

$$\begin{aligned} &f_{11}(u, v) = -c, \quad f_{12}(u, v) = a, \quad g_{11}(u, v) = -v^2, \quad g_{12}(u, v) = 0, \\ &f_{21}(u, v) = 0, \quad f_{22}(u, v) = uv, \quad g_{21}(u, v) = 0, \quad g_{22}(u, v) = -(a+1). \end{aligned} \tag{4.3}$$

Let $w := (u, v) \in X_1$. Using Young's inequality in the form

$$uv \leqslant \varepsilon u^2 + \frac{1}{4\varepsilon} v^2, \qquad u, v \geqslant 0, \quad \varepsilon > 0,$$

a straightforward calculation yields that

$$\begin{aligned} (w, G(w) + A_0 w)_0 &= (u, \Delta u)_0 + (v, \Delta v)_0 + (u, f_1(u, v))_0 + (u, g_1(u, v))_0 \\ &\quad + (u, f_2(u, v))_0 + (v, g_2(u, v))_0 \\ &\leqslant -c \int_\Omega u^2 \, dx - (a+1) \int_\Omega v^2 \, dx + a \int_\Omega uv \, dx + \int_\Omega uv^3 \, dx - \int_\Omega u^2 v^2 \, dx \end{aligned}$$

$$\leqslant \frac{a^2}{4c}\int_\Omega v^2\,\mathrm{d}x - \int_\Omega u^2v^2\,\mathrm{d}x - (a+1)\int_\Omega v^2\,\mathrm{d}x + \int_\Omega uv^3\,\mathrm{d}x$$
$$\leqslant \left(\frac{a^2}{4c} - a - 1\right)\int_\Omega v^2\,\mathrm{d}x + \int_\Omega uv^3\,\mathrm{d}x.$$

Hence, by the hypothesis (4.2), if we choose $w \geqslant 0$ such that

$$|w| \leqslant \sqrt{a+1-\frac{a^2}{4c}},$$

then

$$(w, G(w) + A_0 w)_0 \leqslant 0,$$

provided $\|w\|_{X_1} \leqslant \frac{1}{C_0}\sqrt{(a+1-\frac{a^2}{4c})}$, where $C_0 > 0$ is the constant appearing in (3.3). At the same time, take positive constants η_1 and η_2 satisfying

$$\begin{cases}\eta_1\eta_2 \leqslant a+1,\\ a\eta_2 \leqslant c\eta_1.\end{cases} \tag{4.4}$$

Then, by the decomposition (4.3), the constant vector (η_1, η_2) is a coupled upper bound in relation to (0, 0) of system (4.1).

We are specially interested in the global existence of nonnegative solutions and the stability of equilibria to system (4.1). Now, we apply theorem 3.2 and theorem 3.3 to obtain the following results.

Theorem 4.1. *Let the hypothesis (4.2) hold. Then the following statements are true:*

(1) there exists a unique strong solution $w(t, x)$ to (4.1) defined on the maximal interval of existence, and

(2) the nonnegative solution of (4.1) is global, provided $\|(u_0, v_0)\|_{X_1} \leqslant \frac{1}{C_0}\sqrt{(a+1-\frac{a^2}{4c})}$, where $C_0 > 0$ is the constant appearing in (3.3). Moreover, the trivial solution $w = (0, 0)$ to (4.1) is positively Ljapunov stable.

Example 2. Next we consider the weakly coupled reaction-diffusion system,

$$\begin{cases} u_t - d_1\Delta u = -u\,\mathrm{e}^{u^2} + vl(u) - \lambda u^{2p}v - u^3(u^{2p-2}-1)\,\mathrm{e}^u & \text{in } D,\\ v_t - d_2\Delta v = -v\,\mathrm{e}^{v^2} + ur(v) - v\,\mathrm{e}^{v^2} - \lambda u^3v^2 & \text{in } D,\\ u(t,x) = v(t,x) = 0 & \text{on } S,\\ u(0,x) = u_0(x) \geqslant 0, \quad v(0,x) = v_0(x) \geqslant 0 & \text{in } \Omega,\end{cases} \tag{4.5}$$

where $\lambda > 0$, $p > 1$ are constants and $l, r \in C^3(\mathbb{R}, \mathbb{R})$ such that $0 \leqslant l \leqslant 1$, $0 \leqslant r \leqslant 1$.

Set

$$f_1(u,v) = -u\,\mathrm{e}^{u^2} + vl(u), \qquad f_2(u,v) = -v\,\mathrm{e}^{v^2} + ur(v),$$
$$g_1(u,v) = -\lambda u^{2p}v + u^3(u^{2p-2}-1)\,\mathrm{e}^u, \qquad g_2(u,v) = -v\,\mathrm{e}^{v^2} - \lambda u^3v^2,$$

and write

$$f_{11}(u,v) = -\mathrm{e}^{u^2}, \quad f_{12}(u,v) = l(u), \quad g_{11}(u,v) = u^2(u^{2p-2}-1)\,\mathrm{e}^u, \quad g_{12}(u,v) = -\lambda u^{2p},$$
$$f_{21}(u,v) = r(v), \quad f_{22}(u,v) = -\mathrm{e}^{v^2}, \quad g_{21}(u,v) = -\lambda u^2v^2, \quad g_{22}(u,v) = -\mathrm{e}^{v^2}.$$

Then for each $i = 1, 2$, $f_i(u, v)$ is quasimonotone nondecreasing in R, $g_i(u, v)$ is quasimonotone nonincreasing in R, $f_{11} < 0$, $g_{22} < 0$ and

$$(f_{12}(u,v) + f_{21}(u,v))^2 = (l(u) + r(v))^2 \leqslant 4 \leqslant 4f_{11}(u,v)f_{22}(u,v),$$

for $(u, v) \in \mathbb{R}^2$. Moreover, one can easily check that

$$(g_{12}(u,v) + g_{21}(u,v))^2 \leqslant 4g_{11}(u,v)g_{22}(u,v),$$

provided $|w|$ is sufficiently small, and $(\eta_1, \eta_2) = (1, 1)$ is a coupled upper bound in relation to $(0, 0)$ of system (4.5). Hence it follows from theorem 3.2 and theorem 3.4 that (4.5) has a unique global solution, $w = (u, v)$, satisfying $(u, v) \leqslant (1, 1)$, provided $w_0 = (u_0, v_0) \geqslant 0$ and $\|w_0\|_{X_1}$ is sufficiently small, and the trivial solution $(0, 0)$ is positively Ljapunov stable.

Remark 4.1. A coupled system with polynomial nonlinearities has been studied by Escher and Yin [6]. However, since the nonlinearities in system (4.5) have no monotonicity properties, the stability result established in [6] cannot be applied to system (4.5).

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